THE DIFFERENTIAL CALCULUS AS LANGUAGE

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Abstract. The Differential Calculus provides a precise, standardized language in which to learn about and discuss specific "changes" other than just in mathematics. Unfortunately, rather than to disseminate this language widely, the mathematical community has mostly considered the Differential Calculus as just a part of mathematics, if a fairly central one, and has taught it accordingly. We argue that a major rethinking of whom we want to teach it to and of the way we should then teach it is necessary and would have some far reaching consequences. One might want to learn the language of the Differential Calculus without necessarily wanting to learn its technique as do engineers and scientists or its theory as do mathematicians. To replace the usual Arithmetic and Algebra courses followed by the usual three semesters of Precalculus I-II and Calculus I, we propose, for students who passed or placed out of an integrated Arithmetic-Algebra I sequence with an A, a two four-hour semesters Integrated Precalculus I-II & Calculus I Sequence whose first semester is designed to impart Calculus Literacy and whose second semester is designed as bridge to the mainstream integral calculus. The characteristics of this sequence are that: i. it takes a Lagrangian viewpoint and ii. it proceeds through classes of functions of increasing complexity. This, together with a taskcard format gives realistic access to the Calculus to large numbers of students who never even dreamt of coming anywhere near it.

1. Introduction

If one of the goals of STS is "*to explicate science, technology and their societal relationships*" [2], this immediately raises the questions: To whom and how? There are indeed several problems in trying to define, assess and impart the scientific literacy and, in particular, the mathematical literacy which society must possess if it is to be formally introduced to science and technology. One problem is linguistic in nature and derives from the fact that scientific thought is mostly expressed in technical languages. But a second problem derives from misconceptions current among lay persons about the nature of mathematics and a third problem from misconceptions among mathematicians about what use lay persons have for mathematics.

Any linguistic activity has three components: a language consisting of a vocabulary and of a grammar according to which the sentences in that language are to be constructed, the more or less well specified part of the real world to be described in that language and some kind of relationship between the two through which sentences acquire a meaning and if we place the emphasis on the language, we can define an interpretation of this language

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as a relationship this language can have with the real world. In particular, note that the complexity of a language will tend to be the same as that of the reality which it tries to describe.

In contrast with literary languages, the ways in which technical languages and, in particular, formal languages can be interpreted are fairly well regulated. A first objective for Technical Literacy, then, should be to convey an awareness of what distinguishes technical languages from literary ones. One feature is the very precise meaning given to words. For example, when a master electrician tells his apprentice to go get him a pancake, it doesn't mean that he is hungry! Similarly, in the language of mathematics, there is usually very little room for relativism: Euclid is read today in exactly the same way as when it was written a couple of thousand years ago and "Parallel" still means non-intersecting. Abstraction and generalization, though, raise interpretational issues even in mathematics. For example, "three" does not exist inasmuch as you cannot exhibit it. What you can exhibit is "three-apples" or "three-fingers". As another example, the meaning, and therefore the definition of, addition depends on the nature of the numbers being operated on: whole numbers, signed numbers, fractions, etc. As still another example, the meaning of "parallel" has been extended from the original Euclidean one to serve in other parts of mathematics and, while the essential idea of "non-intersecting" roughly subsists, today the term can involve very subtle notions.

A second problem in trying to impart mathematical literacy to lay persons is that they usually equate, a priori, mathematics with algorithms, that is with mechanical procedures designed to obtain unerringly the answer to some question from given data. This is somewhat paradoxical given the almost complete lack of attention given to this aspect in elementary mathematics education while there certainly is a lot of room for inquiry. The arithmetic algorithms, for instance, are based on usually tacit assumptions. Say three persons stick up the bank of a one-bank town and run away (not too far) with the following: 7 **Chase**, 2 **Cleveland**, 9 **Franklin**, 0 **Hamilton**, 5 **Washington**, 6 **dimes**, 4 **pennies**[1]. Now they want to divide the loot. Clearly, each will only get 2 **Chase**, 0 **Cleveland**, 3 **Franklin**, 0 **Hamilton**, 1 **Washington**, 2 **dimes**, 1 **penny** as they can hardly return to the bank to change the remaining 1 **Chase**, 2 **Cleveland**, 0 **Franklin**, 0 **Hamilton**, 2 **Washington**, 0 **dimes**, 1 **penny** for further division. But this is not the result which the division algorithm would give because it is based on the usually unstated assumption that changing facilities are available. In fact, what little attention has been given to arithmetic algorithms in the name of the New Math has had fairly disastrous effects[2]. At any rate, there are other sides to mathematics that are much more directly relevant to mathematical literacy.

In particular, a very important aspect of Mathematical Literacy, at the Two-year College level, but, we suspect, also at other levels, refers to the contents. Clearly, some arithmetic and algebraic literacy is necessary but then what? The problem is to determine exactly how much or, better yet, what mathematics is indispensable.

[1]In other words, with $72,905.64!

[2]The use of expressions such as "borrowing" and "carrying over" obscures things even more completely.

2. Why the calculus?

In a recent article about what mathematics course, if any, should be universally required from college graduates, the author arrived at the conclusion that none could: "*Assuming that a required mathematics course for everyone is a good idea, algebra is a poor choice. Algebra is important only as a step in the preparation for calculus. As a terminal course applicable to real-life problems, it is virtually worthless. The so-called applications in textbooks are contrived and unrealistic, and are useful only as simplified models of more complex problems found in genuine applications. A universal requirement of calculus would make sense, but that is out of the question.*"[6]

Yet, in Steps Toward a Lean and Lively Calculus, the introductory paper to the 1986 Tulane conference by the same name, R. Douglas started from the premise that: "*Almost all of science is concerned with the study of systems that change, and the study of change is the very heart of the differential calculus*" and, indeed, we believe that the above is predicated on a particular view of the calculus, that expounded in the currently fashionable 10 pound plus texts and we wrote to the author, arguing that "*If we recognize that a 'calculus for the masses' need only be a calculus of analytic functions, about all the algebra that is required to handle them is binomial expansions and long division of polynomials in both ascending and descending powers. The irony, then, is that this Lagrangian calculus, that is by way of Taylor expansions, essentially boils down to rather trivial ... algebra. One might say to simple ... approximate algebra.*" So, the calculus would appear to be both indispensable and feasible. But this raises in turn the question of indispensable for what.

3. Calculus for what?

Originally, of course, the calculus was invented by people who wanted to solve mathematical problems and, if it worked beautifully at first, a rapid escalation ensued and, within a century, the calculus had become inadequate for solving the problems mathematicians were then dealing with. The main reason was that there was a certain fuzziness to the way it was formulated and, also, that it was not systematic enough. In 1797, Lagrange succeeded in setting the calculus on a rigorous footing but it was realized very rapidly that his approach could only deal with a certain kind of functions while the problems which mathematicians wanted to solve involved much more exotic functions. So the search went on until Bolzano and, a few years later, Cauchy and Weierstrass developed the limit approach which could deal satisfactorily with the set of functions that mathematicians were interested in.

Up until the 1920s though, the garden variety calculus, that of Newton and/or Leibniz, continued to be taught as "elementary calculus" even to those interested in solving mathematical problems and then, with the advent of industrialization, to those wanting to solve engineering or physics problems and the limit approach was only taught to future mathematicians as "advanced calculus". But round about that time, the calculus began to be taught by mathematicians who could not do mathematical research and therefore specialized in teaching it and what then happened was that the elementary calculus got to be taught by teaching mathematicians while the advanced calculus remained taught by research mathematicians [3]. This resulted in the teaching mathematicians getting an inferiority complex and eventually deciding to "upgrade" the elementary calculus by using limits too. Engineers and physicists were not too happy with that development and it is from that time that the feud dates between mathematics departments on the one hand and physics and engineering departments on the other. But, by and large, engineers and physicists bit the bullet and, willy-nilly, learned the calculus with limits even if, in their own work, they continued for a long time to cultivate the garden variety.

Finally, about ten years ago, another revolution started to take place. After industrialization, high technology. But, while the industrial revolution had only required that just the comparatively few who were actually going to build, create, invent, ... its artifacts learn the calculus, high technology is much more pervasive and requires that many more than ever before learn the calculus, but this time principally as a communication device. And, indeed, in the Prelude to the Calculus for a New Century 1987 Conference, G. Kolata writes that "*during the past 20 to 30 years. college education has become more commonplace. Now the masses, rather than just the wealthy and the privileged, go to college.*"[5] But this in turn created a new problem and Kolata goes on writing that "*the (students') high school mathematics education frequently is woefully inadequate. To accommodate these students, calculus courses were made less demanding. No longer are students asked to do proofs, for example, only to work out simple problems that are exactly like worked-out examples in their textbooks.*" Similarly, Douglas had deplored that the calculus course had degenerated to "*a watered down, cookbook course in which all (the students) learn are recipes, without even being taught what it is what they are cooking*"[3] The question then remains where to get these greater numbers from and how to make the calculus affordable to them.

4. Calculus Literacy

The mathematical language is usually perceived as an obstacle to comprehension even by textbook writers who profess universally that their writing style yields uniquely crisp exposition, both intuitive and precise, etc. As we noted above, it is not the language which obscures the mathematical reality but the complexity of the mathematical reality itself which forces the complexity of the language. It is, principally, because these authors purport to deal with continuous functions, which are infinitely more complicated than what would at first appear, that they must resort to acrobatic contortions while they try to simultaneously cover their mathematical behind and remain understandable to students of whom they assumed, three pages before, that they did not know what a function is. The irony is that there isn't a single problem in any commercial textbook which involves a continuous function which is not actually analytic[4]. The only way to simplify the language necessary to deal with the calculus is to simplify the mathematical reality to be

[3]The distinction was enshrined in two mathematical associations, the AMS and the MAA.

[4]Roughly speaking, an analytic function is, locally and approximately, like a power function.

described and to acknowledge that we are really only dealing with functions that, locally, are approximately polynomial functions[5].

One of the purposes of a technical language is to crystallize what is important in a subject matter. Unfortunately, importance is a relative concept and what is important to a Real Analyst may be of no importance whatsoever to the rest of us who are interest in practical changes. For example, continuous functions which are not analytic are at most of fractal interest to a lay person. Actually, most of the students at Community College of Philadelphia, and we suspect elsewhere too, do not know what it means for a function to be positive or negative, to have a zero or a pole, to be increasing or decreasing, to have an extreme or to be monotonic, to have an inflection, etc. and can even less determine if a given function has any of these features even when the function is given by a graph! Most of these students, when they look at a wiggly line on a cross, see just that rather than an input-output device and this for a long time in spite of sometimes considerable efforts. Thus, what is important for these students is getting acquainted with the various features which a function can have.

Another useful thing for these students is to realize that there are different viewpoints from which we can look at a function. Given a function, our interest may be punctual as in "*what is its value at this point?*", local as in "*what happens to the output when the input is near* 0 *(very small)? near* ∞ *(very large)*", or global as in "*when is the output positive*".

We then define algebra literacy as that algebra which is indispensable to study functions given by simple algebraic rules, that is polynomial functions, Laurent-polynomial functions and rational functions. While it is true that these functions are seldom found in nature where functions are given by differential equations, they exhibit all the features that the students should become familiar with and therefore provide them with much to experiment with.

In fact, (Laurent-) polynomial functions offer much more than that as they enable us to approximate all usual transcendental functions via their Taylor expansions. This is not just of numerical interest as the Taylor expansions have most of the local qualitative properties of the exact solutions which they approximate. For instance, the addition formulas for the Taylor expansions of e^x, $\cos x$ and $\sin x$ are exactly the same as for the functions themselves but much easier to check and even to discover.

5. Calculus for whom?

If "*we have come to realize that the way we look at the calculus from a disciplinary point of view is not necessarily the way we should look at it when we teach it*" [4], if a broad coalition of organizations has undertaken efforts to revitalize calculus - both in contents and in its teaching, including the National Science Foundation, the Alfred P. Sloan Foundation, the Mathematical Sciences Education Board and the Board of Mathematical Sciences of the National Research Council, and the Mathematical Association of America, and even

[5]The global viewpoint is mostly that of the integral calculus as in "What is the average output between two points"

if the Report of the Content Workshop at the Tulane Conference stated that "*A first semester class is not all engineering students, it is not all physical science majors, it is not all computer sciences majors, it is not all social science majors, and it is definitely not all math majors. The calculus I syllabus is designed for a general audience, 'just plain folks'. (...) Calculus I can be taken as a terminal course-there is no postponement of integration or trigonometry or exponential functions to the second semester*", for all this, it is not entirely clear that these efforts will do much for calculus literacy for the masses. For example, Douglas seems to have had in mind primarily the "*young people studying mathematics, science and engineering*"[6] and early signals from the National Science Foundation would appear to indicate that it is not too interested in funding efforts other than for the mainstream calculus, i.e., "*freshman-sophomore calculus courses or sequences designed as a basis for eventual student access to upper-division mathematics courses*"[1].

For our part, we are particularly interested in the women, minorities and "returning" adults who almost all "*choose not to continue their study of mathematics through calculus, thereby closing career options in mathematics, engineering and the sciences*"[7]. Then, starting from a determination of what the calculus as a means toward technical and scientific literacy should consist of for these students, we should be able to work back to what kind of proficiency in algebra is necessary to acquire it and then back to what kind of proficiency in arithmetic all of this requires. Because these students are initially characterized by an extremely low reading level and a very shaky mathematical preparation, the usual Fractions, Algebra and Word Problems über alles approach, as carried out in commercial textbooks, insists on "mastery" at each level of the conventional progression: Arithmetic, Algebra I, Algebra II, Precalculus I (which it mostly construes as an Algebra II repeat) and Precalculus II (which it mostly construes as Trigonometry). Of course, not many students survive such an extended trial and, in fact, very few even attempt it and thus this approach effectively bars most students from ever dealing significantly with functions, the mathematics of change in any form and thus with any useful literacy. For instance, while, out of 7612 students who enrolled in Algebra I between 1980 and 1985 at Community College of Philadelphia, 2260 (30%) received an A in the course, out of the latter, only 37 (1.6%) ever registered in Calculus I.

In the Calculus über alles approach, one tries to shorten the route to the calculus in as many ways as possible. One is by limiting the required competency in algebra as much as at all possible. Another is by defining the Precalculus as the calculus of functions that are sufficiently simple to be dealt with by elementary algebra but thereby introducing early on the problems for the solution of which the methods of the calculus were invented. A third way is to use Lagrange's approach to the Calculus I proper. We have been developing such an approach and we find that it allows from one quarter to one third of the students enrolled in Calculus I to succeed with an A[7]. Yet, this is still intended to be in the "mainstream" spirit because it presupposes the willingness to invest a considerable amount of time and

[6]This represents about 4% of the total Calculus I enrollment at Community College of Philadelphia.

[7]By way of comparison, in the Anderson-Loftsgaarden study, of the 300,000 students who took mainstream calculus, only 140,000 finished the year with a grade of D or higher.

energy without immediate tangible return, not even that of transfer credit as the tendency is not to get Precalculus transfer credit.

6. An integrated course

What amounts to academic miracle, though, is that the time devoted, say, to arithmetic happens to be exactly the same as that devoted, say, to Precalculus II. The truth of course is that the time slots are predetermined, labeled and then just filled to capacity with some of the contents of some commercially available textbook which then become, ipso facto, indispensable. Another very convenient academic miracle is that requiring "mastery" of these contents allows us to assume that everything will always be at the students' finger tip whenever it will be necessary, even two or three semesters later!

We thus integrated Precalculus I, Precalculus II and Calculus I into a two four-hour semesters sequence. Some of the advantages are that it: i. offers immediate marketability to the students by carrying calculus credit and ii. is low risk in the sense that it does not endanger the cumulative grade point average of reasonable students and/or force them to drop (and thereby lose financial support and self respect) in order to avoid bad grades, iii. is much more efficient in that, at each stage, only what algebra is of immediate use is dealt with. Finally, to bring things to their logical conclusion, we are now contemplating the integration of the arithmetic and the algebra required for the Integrated Precalculus I-II & Calculus I Sequence into one three hour course. We thus propose the following set up:

As a Preparatory course, students would take an Integrated Arithmetic-Algebra course. Less than an A would signal to the students to stop there. Students with an A in the Integrated Arithmetic-Algebra course or with some sort of advanced placement would be allowed and in fact could be urged into the first semester of the Integrated Precalculus I-II & Calculus I Sequence.

The **Differential Calculus I** course is the first semester of the Integrated Precalculus I-II & Calculus I Sequence and the emphasis is Calculus Literacy, that is, literacy in the mathematics of (analytic) change. As such, the course qualifies for "short" calculus credit and reasonable students should expect at least a C. Between two thirds and three quarters of the students should be able to do so. Completion of this semester with a C should be a signal to the students to stop there, a B should be an indication to take the next course as a terminal course and an A that continuation into a mainstream Calculus II should be seriously considered.

The **Differential Calculus II** course is the second semester of the Integrated Precalculus I-II & Calculus I Sequence. Completion of that semester with at least a C completes the equivalent of a regular first semester calculus and allows the student into any regular Calculus II course, not necessarily at Community College of Philadelphia, with the expectation of at least a C. About two thirds of the students in that semester should be able to do so.

The two-semester Differential Calculus sequence is currently being tested at Community College of Philadelphia and at Essex Community College in Baltimore with the partial support of a National Science Foundation Calculus Development grant. Further, large scale, testing is expected under a proposed National Science Foundation Calculus Development grant.

Raising the percentage of the students acceding to Calculus I from 1.6% to, say, 20% would raise the total Calculus I enrollment at Community College of Philadelphia by about 50% and, nationally, would add about 50,000 students a year[8]. This would rapidly create a basis of calculus literate techniciansbut, also, a wider pool from which to recruit tomorrow's engineers and physical, biological and social scientists. It would, if nothing else, broaden the prospects of many students.

7. Instructional strategy

The nature of this student body forces another set of considerations. Lecturing as a means to introduce information is about ruled out for students who can neither sit through a one-hour lecture nor be expected to take notes. Neither can they be expected to read any textbook or to do homework on a systematic basis. Also, a minimum pace must be clearly defined and maintained lest the students, who naturally give in to the most immediate pressure, fall behind. The materials should nevertheless take into account that the students will frequently fall behind. Finally, if the students are not just to memorize and are ever to do any kind of independent thinking, the contents must be highly structured if only to help them keep track of things. To address these issues, we have developed a taskcard format in which each page consists of: a Problem, a couple of lines of Information, a Procedure whereby the problem is worked out step by step and three similar Exercises with answers and space to do the work so that the taskbook contains all of the student's work for the semester. Each taskcard deals with a single concept and each step in the procedure either invokes a previous taskcard as subroutine or, if new, is extremely simple. Each concept is introduced by a taskcard where the function is given by a graph and then, after an Algebra Review card whenever necessary, by a taskcard where the function is given by a rule $f(x) = ...$.

There is much more to transmission of knowledge than to transmission of information. The acquisition of knowledge, which Piaget spent almost half a century investigating, is far from trivial. Knowing that 2 plus 2 is 4 is not the same as knowing how to add and this is not the same as knowing what adding means, etc. The first is fairly easily conveyed and needs mostly be memorized. The second is a bit more tricky as it requires an operational familiarity with syntactic play. The third, as anything which involves a semantic relation, is even trickier since it requires a certain ontological awareness, in this case at least the awareness of the conservation of numbers. Thus, all in all, learning addition involves both the platonic aspect and the formalist aspect of mathematics. This has practical consequences for the efficiency of the learning process proposed to the students. Twenty years ago, Z. P. Dienes observed that playing children spontaneously adhered to the following cycle which he proposed as basis of a learning cycle: First the children play with concrete objects according to given rule and thereby "investigate" ipso facto the concrete effect of

[8]CCP is quite typical of large urban two year colleges and not very different from most others and there are currently about 100,000 Calculus students enrolled in Two-year Colleges.

these rules. Then, the children start changing the rules. At first, the emphasis remains on the objects but after a while the emphasis moves to the investigations of the effects of the changes on the rules and therefore it is the rules themselves which then become the object of the game. And the cycle starts anew with super rules according to which the rules are played with. The children seem to be oscillating between their need for the reassurance given by hard and fast rules and their natural curiosity which seems to be an equally important drive. In the process, the level of abstraction goes up each time one notch.

While our taskcard format could superficially be seen as a form of programmed instruction, our primary goal in developing it was to allow and foster a similar cycle. It acknowledges the students' initial need for hard and fast rules but gives them the basis from which to explore a certain mathematical reality. Once the students have attained a certain level of familiarity with the rules at hand, given the opportunity, many will start investigating and questioning the rules themselves. But exploration is a notoriously dangerous activity and, after a while, and for a while, even these students will want some sort of reassurance. This can take several forms. One is that they can perform adequately on a standard exam. Another is that the structure of the materials also support students' discoveries. Eventually, one would like at least a few of these students to become sensitive to and enjoy the esthetic rewards of consistency. The taskcard format has several additional advantages: it allows a much more individualized pace, it serves as a notebook, it forces the students to pay attention to what they are doing, it allows the instructor to monitor individual progresses, it prepares the students for activities/lectures meant to tie things together. Content architecture. The reason for the success of the taskcard format is really the way in which we organize the taskcard sequence. Our project could not have even been envisioned without a radical restructuring of the contents themselves and without, moreover, a radically different approach to the differential calculus[9].

First we must repeat that, by calculus, we mean here the differential calculus of functions up to and including a few very simple differential equations with the transcendental functions seen as solutions of initial value problems. Moreover, to help keep things in sharp focus, analytic geometry is avoided or recast in functional terms[10]. Specifically, our theme is that, given a function, we investigate a given aspect, sign, variation, optimization or concavity from a given viewpoint: punctual, local or global[11]. Finally, it should go without saying that by no means do we water down the contents. Quite to the contrary, we demand, and get to quite a significant extent, a rather unusual amount of intellectual rigor from the students.

[9]As will be seen of course, neither is really new. It was, in fact, the standard treatment up until the twenties.

[10]For instance, rather than talking about the point-slope formula for a straight line, we prefer to solve the initial value problem: $f"(x) = 0, f'(x_0) = y'_0$ and $f(x_0) = y_0$.

[11]For instance, we found that the students quite appreciate what is involved in the fact that the variation of a function *near* a regular point is given by the sign of its derivative *at* that point. See [8a],[8b] for more details.

The approach which we have recently proposed, [8a], [8b], to carry out this program, essentially that of Lagrange, consists in taking advantage of the fact that, for all practical purposes, the functions encountered in the Calculus are analytic, that is locally and approximately (Laurent) polynomial functions. So, their study reduces, by way of asymptotic *expansions*[12], to that of (possibly shifted) power functions without necessarily involving limits[13]. The expansions are easily obtained by binomial expansions for polynomial functions, division in ascending and descending powers for rational functions and indeterminate coefficients for transcendental functions.

This approach has several advantages: i. It "raises students' conceptual understanding" in that the power functions appear as prototypes and building blocks for "all" other functions[14], ii. it uses, if heavily, very little algebra which "*reduces the demands on students (fresh out of Algebra 1) for traditional manipulations while increasing their need to understand what those manipulations accomplish*", iii. it lends itself perfectly to the use of symbolic calculators which will further help the students concentrate on the calculus rather than being bogged down in diverse technicalities, iv. the amount of rigor can be very easily adjusted to the needs of the students, v. it familiarizes the students with approximations and, as such, prepares them to algorithmic techniques and, last but not least, vi. in Piaget terminology, the students need only be at the concrete operational stage.

The other characteristic of our approach is that we proceed through classes of functions of increasing complexity. We begin with the Precalculus defined as the calculus of those functions that are, essentially, just translated power functions and can therefore be studied globally and by exact algebraic methods, that is Affine Functions ($f(x) = ax + b$), Quadratic Functions ($f(x) = ax^2 + bx + c$) and Homographic Functions ($f(x) = \frac{ax+b}{cx+d}$). We then continue with the Calculus defined as the calculus of functions which are like power functions only locally and approximately , that is (Laurent)-polynomial functions, rational functions and the transcendental functions introduced as solutions of initial value problems but with the usual elementary presentation incorporated. The advantages of such an architecture are that: i. it provides the students with a very strong focus, ii. the subjective level of difficulty faced by the students does not increases appreciably through the sequence. What changes is the nature of the difficulties encountered by the students: at first, the technical difficulties are very small and it is with the concepts themselves that the students have trouble; then, as the students get progressively familiar with the concepts, they are able to shift their attention to technicalities. iii. it ensures, since the problems remain the same throughout, a very strong, persistent reinforcement, iv. it vindicates a very systematic terminology which, in turn, allows us to progressively force the students to document their solutions with textual explanations.

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[12]As distinguished from Taylor *series*.

[13]In fact, limits are easily introduced that way.

[14]For instance, that $f(x) = kx^{2n}$ ($k < 0, n > 0$) is really the prototype of a local maximum by the fact that the expansion at a local maximum will start with kx^{2n}.

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